5.3 Solution Curves / Phase Portraits

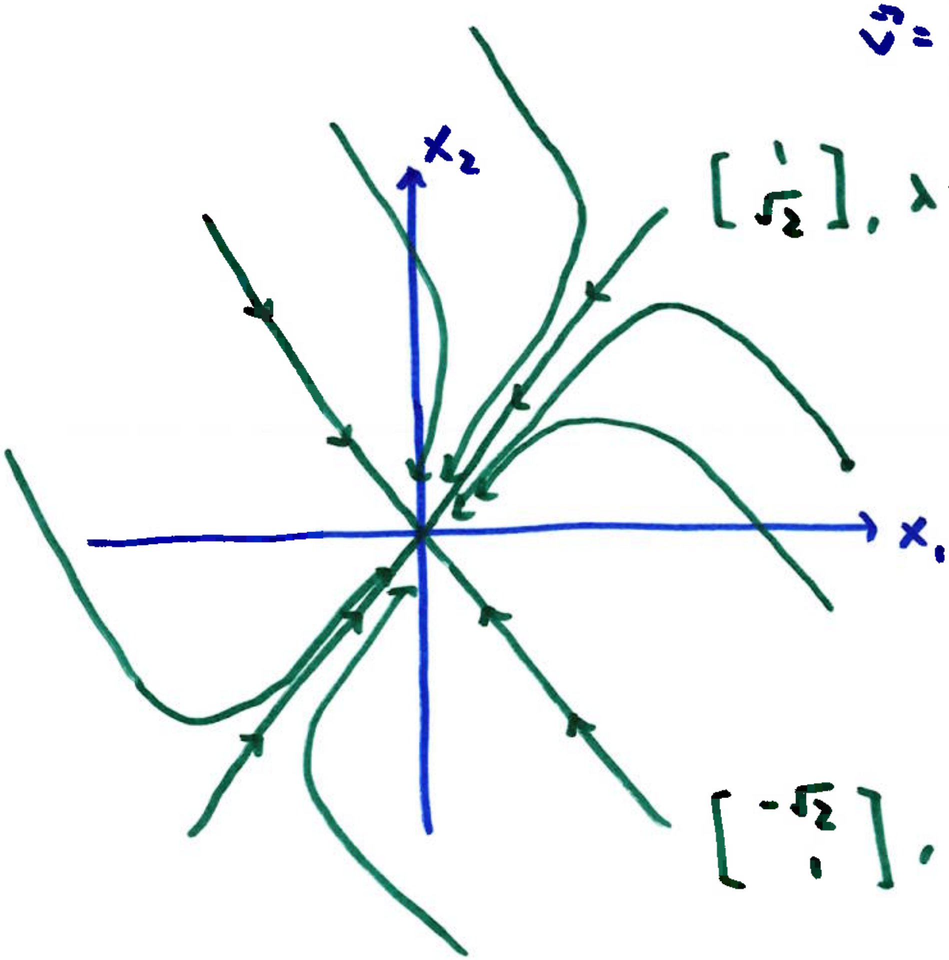
Summary of $\vec{x}' = A\vec{x}$

λ's are real and distinct

$$\vec{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \vec{x}$$

$\lambda = -1, -4$

$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$



$\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \lambda = -1$

$\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}, \lambda = -4$

which one is asymptote into origin?

origin is $t = \infty$ because $e^{\lambda t} \to 0$ as $t \to \infty$

as $t \to \infty$, $e^{-t} > e^{-4t}$

so, the eigenvector w/ $\lambda = -1$ is more important → asymptote

the origin is a

nodal sink

point

solutions go into origin

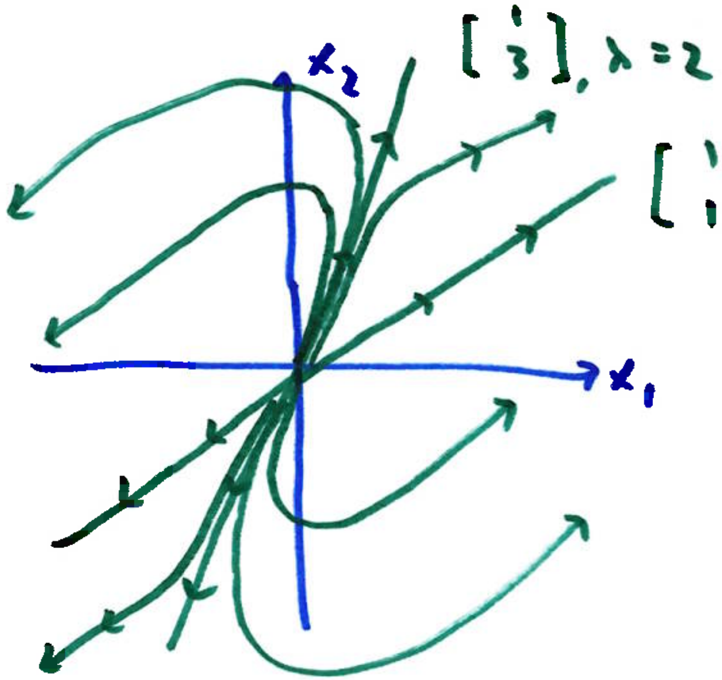
often also called improper nodal sink

follow asymptote into / out of origin

$$\vec{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 2,\ 4$

$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



origin: $t = -\infty$

$e^{+2t} > e^{4t}$

solutions follow $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $(\lambda = 2)$

when they leave origin

origin is an

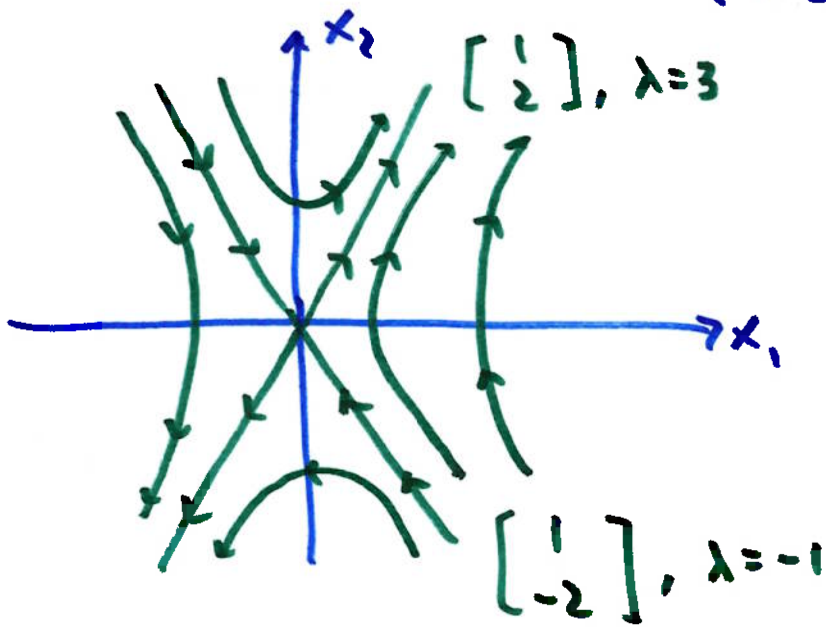
improper nodal source

λ's real and have opposite signs

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

$$\lambda = 3,\ -1$$

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



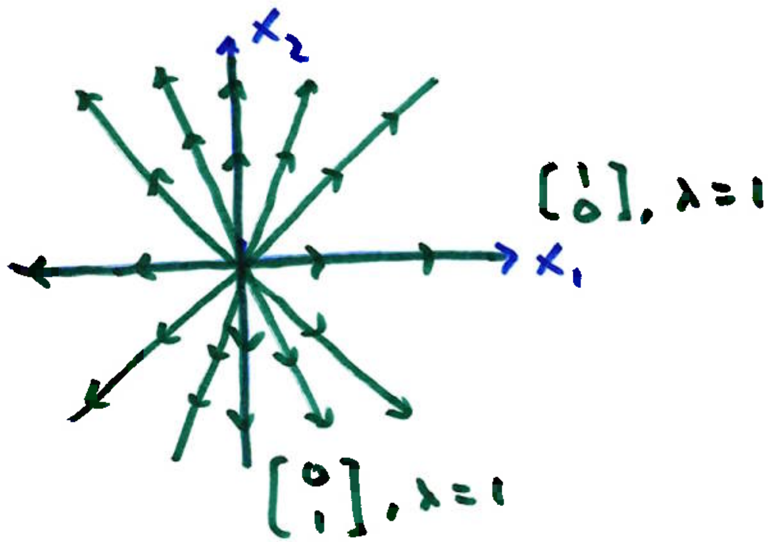
origin is a <u>saddle point</u>

λ's are real and repeated

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ enough eigenvectors



origin is a

proper nodal source

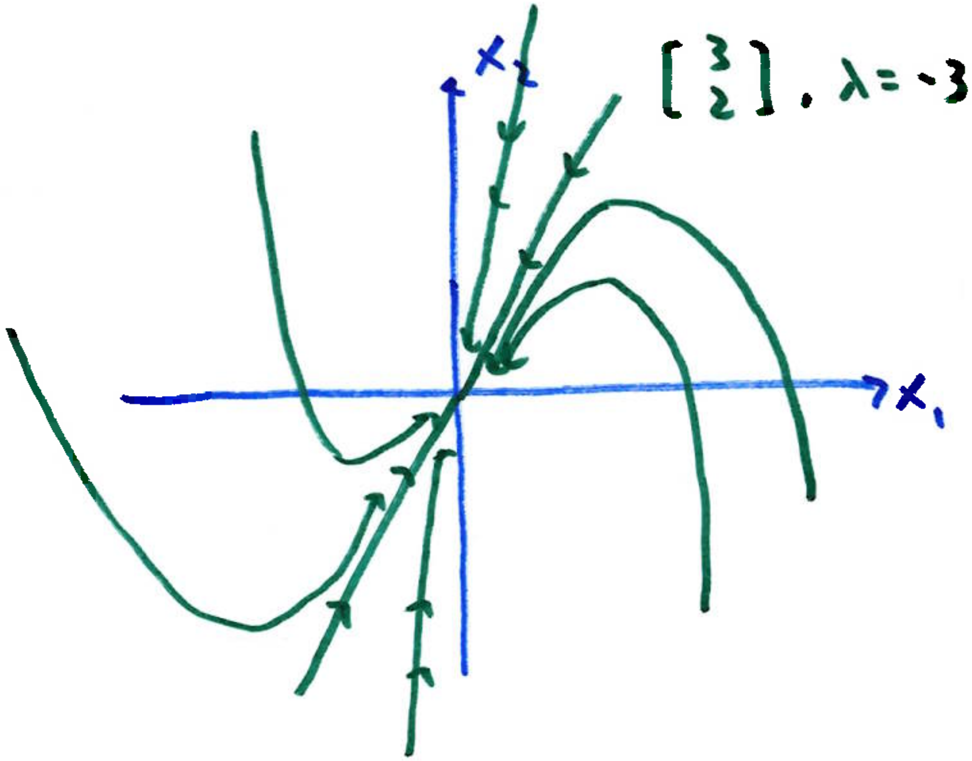
no preferred asymptote

both λ's are the same, so both eigenvectors are equally important → solutions are along linear combos of them

$$\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \vec{x}$$

$\lambda = -3, -3$

$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ defect of one



no 2nd straight line solution visible

"S" shape is typical ("incomplete" spiral)

improper nodal sink

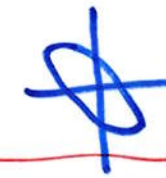
Complex λ's

$$\vec{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\vec{x}$$

$\lambda = i \, , \, -i$

$\vec{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

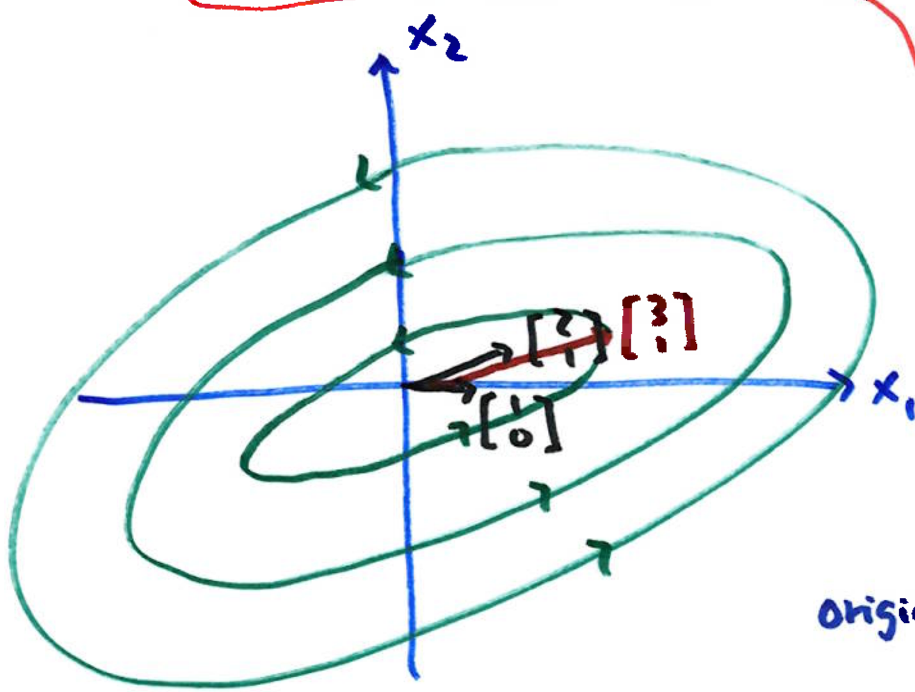
purely imaginary : ovals

how to orient them?  or 

pick an eigenvector: $\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

tell us direction of major axis

* incomplete see next page



direction: pick $\vec{x}$, look at $\vec{x}'$

if $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

right, up

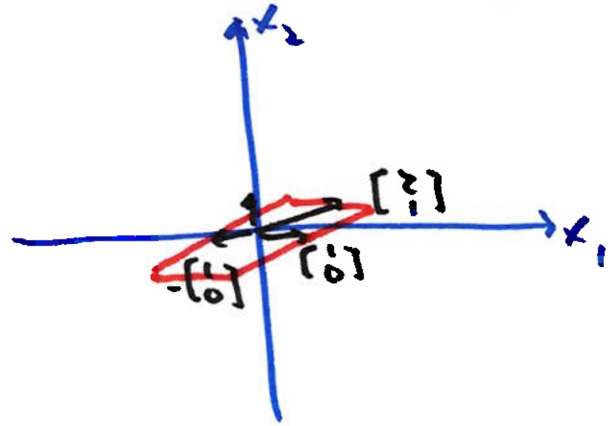
so ccw

origin is a center

ellipse orientation

eigenvectors $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} - i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

form a parallelogram with $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



ellipse is inside the parallelogram

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$

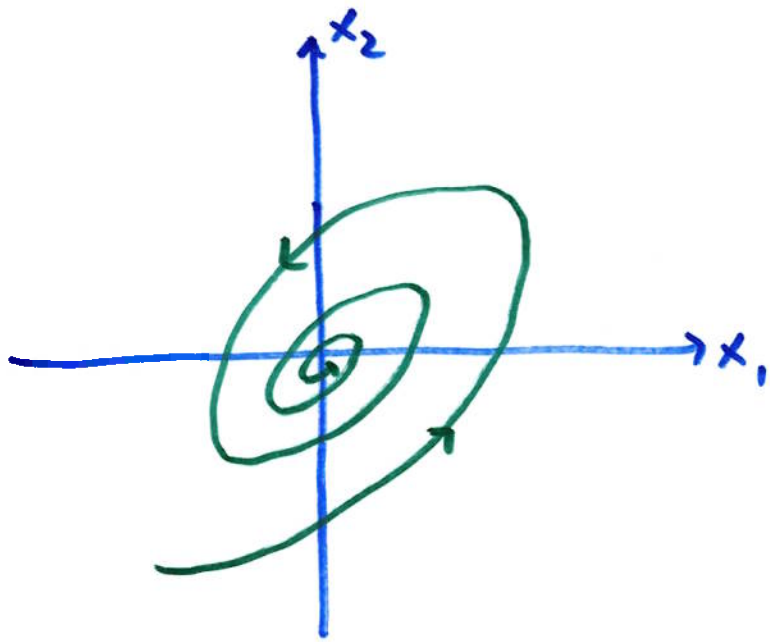
$\lambda = -1+i, \quad -1-i$

$\vec{v} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}, \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

orient the "ovals" the same way as in previous case

this time, real part of λ is -1 so e^{-t} factor drives solutions into origin

ovals shrink as they go



Spiral source/sink

one unusual case: one eigenvalue is 0

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \vec{x} \qquad \lambda = 0,\ 5$$

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

general solution: $\vec{x} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

everything on this line is solution to $\vec{x}' = A\vec{x}$ (a whole line of equilibria)